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Time scale boundary value problems on infinite intervals

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Abstract

Two new existence results are presented for time scale boundary value problems on infinite intervals. The first is based on a growth argument and the second on an upper and lower solution idea. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\mathbb{T}$ (time scale) be a closed subset of $\mathbb{R}$. Define the forward (respectively, backward) jump operator at t for $t < \sup \mathbb{T}$ (respectively, for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t\colon \tau \in \mathbb{T}\} \quad (\text{respectively, } \rho(t) = \sup\{\tau < t\colon \tau \in \mathbb{T}\})$$

for all $t \in \mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$.

If $a < b$ are points in $\mathbb{T}$, then we let

$$[a,b] = \{t \in \mathbb{T}\colon a \leqslant t \leqslant b\}.$$

Throughout this paper $a \in \mathbb{T}$ is fixed, and we assume that there exists $t_n \in \mathbb{T}$, $n \in \{1,2,\ldots\} \equiv \mathbb{N}$, with

$$a < t_1 < t_2 < \cdots < t_n < \cdots \quad \text{with } t_n \uparrow \infty \text{ as } n \to \infty.$$

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Let

$$[a,\infty)=\bigcup_{n=1}^{\infty}[a,t_n].$$

In this paper, we are interested in establishing the existence of solutions to the time scale boundary value problem

$$\begin{aligned}&y^{\Delta\Delta}(t)+f(t,y(\sigma(t)))=0 \quad \text{for } t\in[a,\infty),\\ &y(a)=0,\\ &y(t) \quad \text{is bounded for } t\in[a,\infty),\end{aligned} \tag{1.1}$$

where $a\in\mathbb{T}$ is fixed. To understand (1.1) we need to recall some standard definitions (see [1,2,7–11] for an introduction to this subject).

Definition 1.1. Fix $t\in\mathbb{T}$. Let $y:\ \mathbb{T}\to\mathbb{R}$. Then we define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\varepsilon>0$ there is a neighborhood U of t with

$$|[y(\sigma(t))-y(s)]-y^{\Delta}(t)[\sigma(t)-s]|<\varepsilon|\sigma(t)-s|$$

for all $s\in U$. We call $y^{\Delta}(t)$ the derivative of $y(t)$.

Definition 1.2. If $F^{\Delta}(t)=f(t)$ then we define the integral by

$$\int_a^t f(\tau)\Delta\tau=F(t)-F(a).$$

Time scale boundary value problems on the finite interval have received a lot of attention in the literature (see [2,4–6,8,9]). However, to our knowledge, this is the first paper discussing time scale boundary value problems on the infinite interval. Two existence results will be presented, one based on a nonlinear alternative of Leray–Schauder type [3] and the other on Schauder's fixed point theorem [3]. For the convenience of the reader, we state these two results here.

Theorem 1.1. *Let C be a convex subset of a Banach space E, and let U be an open subset of C with $0\in U$. Then every compact, continuous map $N:\bar{U}\to C$ has at least one of the following two properties*:

(A1) *N has a fixed point in $\bar{U}$*; *or*
(A2) *there is an $x\in\partial U$ and $\lambda\in(0,1)$ with $x=\lambda Nx$.*

Theorem 1.2. *Let K be a convex subset of a Banach space E and $N:K\to K$ a compact, continuous map. Then N has a fixed point in K.*

2. Existence

Let $a \in \mathbb{T}$. Consider the time scale boundary value problem

$$\begin{aligned} &y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, \infty), \\ &y(a) = 0, \\ &y(t) \quad \text{is bounded for } t \in [a, \infty). \end{aligned} \tag{2.1}$$

Throughout this section we assume that there exists $t_n \in \mathbb{T}$, $n \in \mathbb{N}$, with

$$a < t_1 < t_2 < \cdots < t_n < \cdots \quad \text{with } t_n \uparrow \infty \text{ as } n \to \infty.$$

Two existence results will be established in this section, one based on a growth argument and the other based on the idea of upper and lower solutions.

Theorem 2.1. *Suppose the following conditions are satisfied*:

$$f : [a, \infty) \times \mathbb{R} \to \mathbb{R} \quad \textit{is continuous}, \tag{2.2}$$

$$\begin{aligned} &|f(t, u)| \leqslant \phi(t) w(|u|) \quad \textit{on } [a, \infty) \times \mathbb{R} \textit{ with } w \geqslant 0 \textit{ continuous} \\ &\quad \textit{and nondecreasing on } [0, \infty) \textit{ and } \phi : [a, \infty) \to [0, \infty) \textit{ continuous} \end{aligned} \tag{2.3}$$

and

$$\exists r > 0 \textit{ with } \quad \frac{r}{w(r) \sup_{n \in \mathbb{N}} [\sup_{t \in [a, \sigma^2(t_n)]} \int_a^{\sigma(t_n)} \phi(s) |G_n(t, s)| \Delta s]} > 1; \tag{2.4}$$

here $G_n(t, s)$ is Green's function (*see* [8,9]) *for*

$$\begin{aligned} &y^{\Delta\Delta}(t) = 0 \quad \textit{for } t \in [a, t_n], \\ &y(a) = 0, \\ &y(\sigma^2(t_n)) = 0. \end{aligned} \tag{2.5}$$

Then (2.1) *has a solution $y \in C[a, \infty)$ with $|y(t)| \leqslant r$ for $t \in [a, \infty)$.*

Remark 2.1. By $C[a, \infty)$ we mean the space of continuous functions $u : [a, \infty) \to \mathbb{R}$.

Proof. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$\begin{aligned} &y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_n], \\ &y(a) = 0, \\ &y(\sigma^2(t_n)) = 0. \end{aligned} \tag*{(2.6)^n}$$

We begin by showing that $(2.6)^n$ has a solution $y_n \in C[a, \sigma^2(t_n)]$ with $|y_n(t)| < r$ for $t \in [a, \sigma^2(t_n)]$ (note also that $y_n^{\Delta\Delta} \in C_{rd}[a, t_n]$ since $y_n^\sigma = y_n \circ \sigma$ is rd-continuous on $[a, t_n]$).

Remark 2.2. By $C[a,\sigma^2(t_n)]$ we mean the Banach space of continuous functions $y:[a,\sigma^2(t_n)]\to\mathbb{R}$ equipped with the norm $|y|_n=\sup_{t\in[a,\sigma^2(t_n)]}|y(t)|$. $C_{rd}[a,t_n]$ is the space of rd-continuous functions [11] on $[a,b]$.

To see the above we will use Theorem 1.1, so we consider the boundary value problem

$$\begin{aligned} &y^{\Delta\Delta}(t)+\lambda f(t,y(\sigma(t)))=0 \quad \text{for } t\in[a,t_n],\\ &y(a)=0,\\ &y(\sigma^2(t_n))=0 \end{aligned} \tag{2.7$_\lambda^n$}$$

for $0<\lambda<1$. Solving $(2.7)_\lambda^n$ is equivalent [9] to solving the fixed point problem $y=\lambda Ny$ where $N:C[a,\sigma^2(t_n)]\to C[a,\sigma^2(t_n)]$ is given by

$$Ny(t)=\int_a^{\sigma(t_n)} G_n(t,s)f(s,y(\sigma(s)))\Delta s.$$

In [4] we showed that

$$N:C[a,\sigma^2(t_n)]\to C[a,\sigma^2(t_n)] \quad \text{is continuous and completely continuous.} \tag{2.8}$$

Next, let y be any solution of $y=\lambda Ny$ for $0<\lambda<1$. Then, for $t\in[a,\sigma^2(t_n)]$ we have from [10, p. 38] or [11, p. 35] that

$$\begin{aligned} |y(t)| &\leqslant \int_a^{\sigma(t_n)} |G_n(t,s)|\phi(s)w(|y(\sigma(s))|)\Delta s\\ &\leqslant w(|y|_n)\int_a^{\sigma(t_n)} |G_n(t,s)|\phi(s)\Delta s\\ &\leqslant w(|y|_n)\sup_{n\in\mathbb{N}}\left[\sup_{t\in[a,\sigma^2(t_n)]}\int_a^{\sigma(t_n)}\phi(s)|G_n(t,s)|\Delta s\right], \end{aligned}$$

where $|y|_n=\sup_{t\in[a,\sigma^2(t_n)]}|y(t)|$. Consequently,

$$\frac{|y|_n}{w(|y|_n)\sup_{n\in\mathbb{N}}[\sup_{t\in[a,\sigma^2(t_n)]}\int_a^{\sigma(t_n)}\phi(s)\,|G_n(t,s)|\Delta s]}\leqslant 1. \tag{2.9}$$

Now (2.4) and (2.9) imply $|y|_n\neq r$. Let

$$E=C[a,\sigma^2(t_n)] \quad \text{and} \quad U=\{u\in C[a,\sigma^2(t_n)]:\ |u|_n<r\}.$$

Now Theorem 1.1 (notice (A2) does not hold since any solution of $y=\lambda Ny$ with $0<\lambda<1$ satisfies $|y|_n\neq r$) guarantees that N has a fixed point $y_n\in\bar{U}$, i.e. $(2.6)^n$ has a solution $y_n\in C[a,\sigma^2(t_n)]$ with $|y_n|_n<r$ (note $|y_n|_n\leqslant r$ by Theorem 1.1 but $|y_n|_n\neq r$ by an argument similar to the one above). We can use this argument for each $n\in\mathbb{N}$.

For $k\in\mathbb{N}$ let

$$u_k(t)=\begin{cases} y_k(t), & t\in[a,\sigma^2(t_k)],\\ 0, & t\in[\sigma^2(t_k),\infty).\end{cases}$$

Let $S=\{u_k\}_{k=1}^{\infty}$. Notice that

$$|u_k(x)|<r \quad \text{for } x\in[a,\sigma^2(t_1)],\quad k\in\mathbb{N}.$$

Also for $k\in\mathbb{N}$ and $t\in[a,\sigma^2(t_1)]$ we have

$$u_k(t)=\int_a^{\sigma(t_1)} G_1(t,s)f(s,u_k(\sigma(s)))\Delta s+t\frac{u_k(\sigma^2(t_1))}{\sigma^2(t_1)}.$$

Thus, for $k\in\mathbb{N}$ and $t,x\in[a,\sigma^2(t_1)]$ we have

$$u_k(t)-u_k(x)=\int_a^{\sigma(t_1)} [G_1(t,s)-G_1(x,s)]f(s,u_k(\sigma(s)))\Delta s+(t-x)\frac{u_k(\sigma^2(t_1))}{\sigma^2(t_1)}$$

and this together with (2.3) and [11, p. 35] yields

$$|u_k(t)-u_k(x)|\leqslant w(r)\int_a^{\sigma(t_1)} |G_1(t,s)-G_1(x,s)|\phi(s)\Delta s+|t-x|\frac{r}{\sigma^2(t_1)}.$$

The Arzelà–Ascoli Theorem [12] guarantees that there is a subsequence $N_1^\star$ of $\mathbb{N}$ and a function $z_1\in C[a,\sigma^2(t_1)]$ with $u_k\to z_1$ in $C[a,\sigma^2(t_1)]$ as $k\to\infty$ through $N_1^\star$. Let $N_1=N_1^\star\setminus\{1\}$. Notice that

$$|u_k(x)|<r \quad \text{for } x\in[a,\sigma^2(t_2)],\quad k\in N_1.$$

Also for $k\in N_1$ and $t,x\in[a,\sigma^2(t_2)]$ we have

$$|u_k(t)-u_k(x)|\leqslant w(r)\int_a^{\sigma(t_2)} |G_2(t,s)-G_2(x,s)|\phi(s)\Delta s+|t-x|\frac{r}{\sigma^2(t_2)}.$$

The Arzelà–Ascoli Theorem guarantees that there is a subsequence $N_2^\star$ of N_1 and a function $z_2\in C[a,\sigma^2(t_2)]$ with $u_k\to z_2$ in $C[a,\sigma^2(t_2)]$ as $k\to\infty$ through $N_2^\star$. Note that $z_1=z_2$ on $[a,\sigma^2(t_1)]$ since $N_2^\star\subseteq N_1$. Let $N_2=N_2^\star\setminus\{2\}$. Proceed inductively to obtain for $m\in\{3,4,\ldots\}$ a subsequence $N_m^\star$ of N_{m-1} and a function $z_m\in C[a,\sigma^2(t_m)]$ with $u_k\to z_m$ in $C[a,\sigma^2(t_m)]$ as $k\to\infty$ through $N_m^\star$. Then let $N_m=N_m^\star\setminus\{m\}$.

Define a function y as follows. Fix $x\in(a,\infty)$ and let $m\in\mathbb{N}$ with $x\leqslant\sigma^2(t_m)$. Then define $y(x)=z_m(x)$. Then $y\in C[a,\infty)$, $y(a)=0$ and $|y(x)|\leqslant r$ for $x\in[a,\infty)$. Again fix $x\in(a,\infty)$ and let $m\in\mathbb{N}$ with $x\leqslant\sigma^2(t_m)$. Then for $n\in N_m$ we have

$$u_n(x)=\int_a^{\sigma(t_m)} G_m(x,s)f(s,u_n(\sigma(s)))\Delta s+x\frac{u_n(\sigma^2(t_m))}{\sigma^2(t_m)}.$$

Let $n \to \infty$ through N_m (using [10, p. 38] or [11, p. 35]) to obtain

$$z_m(x) = \int_a^{\sigma(t_m)} G_m(x,s) f(s, z_m(\sigma(s)))\Delta s + x\frac{z_m(\sigma^2(t_m))}{\sigma^2(t_m)},$$

i.e.

$$y(x) = \int_a^{\sigma(t_m)} G_m(x,s) f(s, y(\sigma(s)))\Delta s + x\frac{y(\sigma^2(t_m))}{\sigma^2(t_m)}.$$

We can use this argument for each $x \in [a, \sigma^2(t_m)]$, and for each $m \in \mathbb{N}$. Thus

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_m]$$

for each $m \in \mathbb{N}$. $\square$

Remark 2.3. It is easy to check whether the requirement that $w : [0,\infty) \to [0,\infty)$ (as described in (2.3)) be nondecreasing can be omitted, provided (2.4) is replaced by

$$\exists r > 0 \text{ with } \quad \frac{r}{[\sup_{z\in[0,r]} w(z)] \sup_{n\in\mathbb{N}} [\sup_{t\in[a,\sigma^2(t_n)]} \int_a^{\sigma(t_n)} \phi(s)|G_n(t,s)|\Delta s]} > 1.$$

Remark 2.4. We note that it is easy to establish the analogue of Theorem 2.1 for the boundary value problem

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0, \quad \text{for } t \in [a,\infty),$$
$$\alpha y(a) - \beta y^{\Delta}(a) = \gamma, \quad \alpha \geqslant 0,\ \beta \geqslant 0 \quad \text{with } \alpha^2 + \beta^2 > 0,$$
$$y(t) \quad \text{is bounded for } t \in [a,\infty).$$

In this case, the function $G_n(t,s)$ in the analogue of (2.4) is Green's function for

$$y^{\Delta\Delta}(t) = 0 \quad \text{for } t \in [a, t_n],$$
$$\alpha y(a) - \beta y^{\Delta}(a) = 0,$$
$$y(\sigma^2(t_n)) = 0.$$

Next we present an existence result based on the notion of upper and lower solutions. By an upper solution β to (2.1) we mean a function $\beta \in C[a,\infty)$ with

$$\begin{aligned} &\beta^{\Delta\Delta}(t) + f(t, \beta(\sigma(t))) \leqslant 0 \quad \text{for } t \in [a,\infty), \\ &\beta(a) \geqslant 0, \\ &\beta(t) \quad \text{is bounded for } t \in [a,\infty), \end{aligned} \tag{2.10}$$

and by a lower solution α to (2.1) we mean a function $\alpha \in C[a,\infty)$ with

$$\begin{aligned} &\alpha^{\Delta\Delta}(t) + f(t, \alpha(\sigma(t))) \geqslant 0 \quad \text{for } t \in [a,\infty), \\ &\alpha(a) \leqslant 0, \\ &\alpha(t) \quad \text{is bounded for } t \in [a,\infty). \end{aligned} \tag{2.11}$$

Theorem 2.2. *Suppose* (2.2) *holds and there exists* α, β *respectively lower and upper solutions to* (2.1) *with* $\alpha(t) \leqslant \beta(t)$ *for* $t \in [a, \infty)$. *Then* (2.1) *has a solution* $y \in C[a, \infty)$ *with* $\alpha(t) \leqslant y(t) \leqslant \beta(t)$ *for* $t \in [a, \infty)$.

Proof. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$\begin{aligned} &y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_n], \\ &y(a) = 0, \\ &y(\sigma^2(t_n)) = \frac{\alpha(\sigma^2(t_n)) + \beta(\sigma^2(t_n))}{2}. \end{aligned} \tag{2.12^n}$$

We first show $(2.12)^n$ has a solution $y_n \in C[a, \sigma^2(t_n)]$ with $\alpha(t) \leqslant y_n(t) \leqslant \beta(t)$ for $t \in [a, \sigma^2(t_n)]$. To see this we use Theorem 1.2, so the idea is to look at the boundary value problem

$$\begin{aligned} &y^{\Delta\Delta}(t) + f^{\star}(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_n], \\ &y(a) = 0, \\ &y(\sigma^2(t_n)) = \frac{\alpha(\sigma^2(t_n)) + \beta(\sigma^2(t_n))}{2}, \end{aligned} \tag{2.13^n}$$

where

$$f^{\star}(t,u) = \begin{cases} f(t, \beta(\sigma(t))) + \dfrac{u - \beta(\sigma(t))}{1 + |u|} & \text{if } u \geqslant \beta(\sigma(t)), \\ f(t,u) & \text{if } \alpha(\sigma(t)) \leqslant u \leqslant \beta(\sigma(t)), \\ f(t, \alpha(\sigma(t))) + \dfrac{u - \alpha(\sigma(t))}{1 + |u|} & \text{if } u \leqslant \alpha(\sigma(t)). \end{cases}$$

Solving $(2.13)^n$ is equivalent to solving the fixed point problem $y = Ny$ where $N : C[a, \sigma^2(t_n)] \to C[a, \sigma^2(t_n)]$ is given by

$$Ny(t) = h_n(t) + \int_a^{\sigma(t_n)} G_n(t,s) f(s, y(\sigma(s)))\Delta s,$$

where G_n is as described in (2.5) and h_n is the solution of

$$\begin{aligned} &y^{\Delta\Delta}(t) = 0 \quad \text{for } t \in [a, t_n], \\ &y(a) = 0, \\ &y(\sigma^2(t_n)) = \frac{\alpha(\sigma^2(t_n)) + \beta(\sigma^2(t_n))}{2}. \end{aligned} \tag{2.14}$$

It is easy to see [4] (note $f^{\star}$ is bounded) that

$$N : C[a, \sigma^2(t_n)] \to C[a, \sigma^2(t_n)] \quad \text{is continuous and compact.} \tag{2.15}$$

Now, Theorem 1.2 guarantees that $(2.13)^n$ has a solution $y_n \in C[a, \sigma^2(t_n)]$. In fact, the argument in [6] guarantees that

$$\alpha(t) \leqslant y_n(t) \leqslant \beta(t) \quad \text{for } t \in [a, \sigma^2(t_n)]. \tag{2.16}$$

Consequently, y_n is a solution of $(2.12)^n$ with (2.16) holding.

For $k \in \mathbb{N}$ let

$$u_k(t) = \begin{cases} y_k(t), & t \in [a, \sigma^2(t_k)], \\ y_k(\sigma^2(t_k)), & t \in [\sigma^2(t_k), \infty). \end{cases}$$

Proceed inductively as in Theorem 2.1 to obtain for $m \in \{1, 2, \ldots\}$ a subsequence $N_m^\star$ of N_{m-1} (here $N_0 = \mathbb{N}$) and a function $z_m \in C[a, \sigma^2(t_m)]$ with $u_k \to z_m$ in $C[a, \sigma^2(t_m)]$ as $k \to \infty$ through $N_m^\star$. Then let $N_m = N_m^\star \setminus \{m\}$.

Define a function y as follows. Fix $x \in (a, \infty)$ and let $m \in \mathbb{N}$ with $x \leqslant \sigma^2(t_m)$. Then define $y(x) = z_m(x)$. Then $y \in C[a, \infty)$, $y(a) = 0$ and $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ for $x \in [a, \infty)$. Essentially the same reasoning as in Theorem 2.1 guarantees that

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_m]$$

for each $m \in \mathbb{N}$. $\square$

Remark 2.5. The ideas in Theorem 2.2 extend to the boundary value problem

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, \infty),$$

$$y(a) = \gamma,$$

$$y(t) \quad \text{is bounded for } t \in [a, \infty).$$

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